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Periodic Solutions of a Newtonian Equation: Stability by the Third Approximation

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This paper presents a sufficient condition for the stability of periodic solutions of a newtonian equation. This condition depends on the third order approximation and does not involve small parameters. An application to an equation with cubic potential is given.

1. INTRODUCTION

This paper studies the stability in the sense of Lyapunov of periodic solutions of the scalar differential equation

$$x'' = f(t, x) \tag{1.1}$$

where f is periodic in time (T is the period). This stability problem has a nonlinear character and the first Lyapunov method cannot be applied. Given a T-periodic solution θ that is elliptic and satisfies certain additional conditions, there is a recursive procedure which transforms the equation to a hamiltonian system of the form

$$z' = 2i\,\partial_{\bar{z}} H(t, z, \bar{z})$$

with $z = q + ip$, $H = \omega/2T\,|z|^2 + \beta_1/4T\,|z|^4 + \cdots + \beta_{n-1}/2nT\,|z|^{2n} + r(t, z, \bar{z})$. The coefficients ω, $\beta_1, ..., \beta_{n-1}$ are real and r is a T-periodic remainder of order $o(|z|^{2n})$, see [[1], appendice 7]. This is the Birkhoff Normal Form and the numbers $\beta_1, ..., \beta_{n-1}, ...$ are the so called twist coefficients. They depend on the derivatives of f (evaluated at θ) up to the order $2n-1$ and, when some of them is different from zero θ is stable. This is a consequence of the Twist Theorem of Moser [16, 25]. In addition, the abstract theory of twist mappings can be applied to describe the dynamics of the equation in a neighborhood of θ and to prove the existence of infinitely many subharmonic solutions [9, 18, 13].

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After translating θ to the origin, the equation is expressed in the form

$$x'' + a(t)\,x + b(t)\,x^2 + c(t)\,x^3 + \cdots = 0 \tag{1.2}$$

where a, b, c and the remaining terms are periodic functions of the same period. The reason for writing the equation in this way is that the stability is generically decided by the third approximation. In fact $\beta_1 \neq 0$ in most cases.

The twist coefficients are classically employed in the study of small perturbations of autonomous equations. To illustrate this situation consider the equation

$$x'' + a_\varepsilon(t)\,x + b_\varepsilon(t)\,x^2 + c_\varepsilon(t)\,x^3 + \cdots = 0$$

where ε is a small parameter and a_ε, b_ε and c_ε are functions that depend continuously of ε and are constant for $\varepsilon = 0$. This equation admits the equilibrium $x = 0$ and it is assumed that for $\varepsilon = 0$ this equilibrium is stable and has not strong resonances. Then $\beta_1 = \beta_1(\varepsilon)$ is continuous with respect to ε and it is computable for $\varepsilon = 0$. If $\beta_1(0) \neq 0$ it is clear that $x = 0$ is stable for small values of ε.

In this paper the first twist coefficient is used to obtain stability results that are not based on the method of the small parameter. The main result of the paper refers to equation (1.2) and says that the equilibrium is stable ($\beta_1 \neq 0$) when the linearized equation $y'' + a(t)\,y = 0$ satisfies certain conditions that will be precised later and the coefficients b and c have constant sign ($b \leqslant 0$ or $b \geqslant 0$; $c \leqslant 0$). These conditions do not impose restrictions on the variation of b and c and this makes possible to deal with some nonlocal periodic problems. Two examples to which the main result can be applied are the equations

$$x'' + \lambda x + e^{-x} = p(t), \qquad 0 < \lambda \leqslant \lambda_0 \tag{1.3}$$

and

$$x'' + x^2 = p(t), \qquad \int_0^T p(t)\,dt \leqslant p_0, \tag{1.4}$$

where λ_0 and p_0 are certain constants that will be determined later and satisfy the estimates

$$\left(\frac{\pi}{2T}\right)^2 < \lambda_0 < \left(\frac{2\pi}{3T}\right)^2$$

and

$$\tfrac{1}{4} < T^3 p_0 < \tfrac{64}{81}.$$

In both cases p is a T-periodic function (possibly far away from constant functions). A combination of degree theory and the main result of this paper allows us to conclude that, for equations (1.3) and (1.4), one of the alternatives below holds:

(o) Every solution is unbounded

(i) There exists a unique T-periodic solution that is unstable. Moreover, every solution that is bounded in the future is asymptotic to it.

(ii) There exist exactly two T-periodic solutions, one of them stable and another unstable. Moreover, the stable solution is surrounded by infinitely many subharmonic and quasi-periodic solutions.

To conclude the exposition of results it must be noticed that this paper is a continuation of [20]. The main result can be seen as a sharp version of Theorem 1 in [20] and equation (1.3) was already studied in the previous paper assuming the more conservative estimate $0 < \lambda \leqslant (\pi/3T)^2$. The study of (1.4) involves new ideas and the results in the previous paper cannot be employed.

The rest of the paper is organized as follows. In Section 2 we collect some known results on Normal Forms that will be very useful later. Instead of working with the Birkhoff Normal Form of a periodic hamiltonian system as described above, we consider Normal Forms of area-preserving mappings. This approach is simpler and leads to the same conclusions via the Poincaré map. In Section 3 we introduce the concept of twist periodic solution and state the main theorem of the paper. An elliptic solution without strong resonances and such that $\beta_1 \neq 0$ is of twist type. However, this is not the exact definition because resonance at the fourth root of unity will also be allowed. After the main result we prove a corollary that resembles the classical criterion of Lyapunov for stability of Hill's equation. The complete proof of the main theorem is presented in Section 4. Finally, Section 5 is devoted to applying the previous results to equation (1.4). We do not discuss the results for (1.3) because the proofs do not involve any idea beyond those in [20].

2. The Normal Form of Degree Three of an Elliptic Fixed Point

Let $D \subset \mathbb{C}$ be a disk centered at the origin and denote by $A^p = A^p(D)$, $p = 1, ..., \infty$ the class of mappings

$$F\colon D \subset \mathbb{C} \to \mathbb{C}, \qquad F = F(z, \bar{z})$$

satisfying

(i) $z = 0$ is a fixed point of F

(ii) $F \in C^p(D, \mathbb{C})$

(iii) $|\partial_z F|^2 - |\partial_{\bar{z}} F|^2 = 1$ in D (area-preserving condition).

Given $F \in A^1$, the differential of F at the origin (denoted by $dF(0)$) belongs to the symplectic group $Sp(\mathbb{R}^2)$ and the corresponding eigenvalues λ_1, λ_2 satisfy $\lambda_1 \lambda_2 = 1$. When these eigenvalues are not real and have modulus one the fixed point $z = 0$ is said to be elliptic. In such case $dF(0)$ is conjugate in $Sp(\mathbb{R}^2)$ to the rotation $R_\lambda(z, \bar{z}) = \lambda z$ where λ coincides with one of the eigenvalues. The map R_λ can be seen as a normal form of degree one. Next we introduce the normal form of degree three.

PROPOSITION 2.1. *Assume that $F \in A^3$ and $dF(0)$ is conjugate to R_λ in $Sp(\mathbb{R}^2)$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $\lambda \neq \pm 1$. Then there exists $\Psi \in A^\infty$ such that the conjugate map $N = \Psi^{-1} \circ F \circ \Psi$ has an expansion as described below in each case*:

Case I. $\lambda^n \neq 1$, $n = 1, 2, 3, 4$, $N(z, \bar{z}) = \lambda[z + i\beta\, |z|^2 z + \cdots]$

Case II. $\lambda = \pm i$, $N(z, \bar{z}) = \lambda[z + i\beta\, |z|^2 z + \gamma \bar{z}^3 \cdots]$

Case III. $\lambda = \pm\omega$, $\omega = \exp(2\pi i/3)$, $N(z, \bar{z}) = \lambda[z + l\bar{z}^2 + i\beta\, |z|^2 z + \cdots]$.

The dots denote a remaining term of order $o(|z|^3)$, as $z \to 0$, and β, γ, l are constants with $\beta \in \mathbb{R}$ and $\gamma, l \in \mathbb{C}$.

The proof of this result is obtained using the methods described in [[25], Sect. 23]. Some additional details can be seen in [10]. The Taylor expansion of N up to degree three is called the normal form of degree three. In case I this normal form is uniquely determined (see [8] for a proof) while in cases II and III it is unique up to conjugation by a rotation. These facts imply that the coefficients β, $|\gamma|$, $|l|$ are uniquely determined. Sometimes we shall make explicit the dependence of these constants with respect to F and write $\beta = \beta(F)$,... Next result shows that when F is sufficiently smooth the normal form of degree 3 decides in most cases the stability of the origin.

PROPOSITION 2.2. *In the assumptions of Proposition 2.1 and assuming in addition that $F \in A^5$, the fixed point $z = 0$ is Lyapunov stable if one of the following alternatives holds*:

$$\lambda^n \neq 1, \qquad n = 1, 2, 3, 4 \qquad \textit{and} \qquad \beta \neq 0, \tag{2.1}$$

$$\lambda = \pm i \qquad \textit{and} \qquad |\beta| > |\gamma|, \tag{2.2}$$

$$\lambda = \pm\omega \qquad \textit{and} \qquad l = 0, \qquad \beta \neq 0. \tag{2.3}$$

The fixed point is unstable if one of the alternatives below holds:

$$\lambda = \pm i \qquad \text{and} \qquad |\beta| < |\gamma|, \tag{2.4}$$

$$\lambda = \pm \omega \qquad \text{and} \qquad l \neq 0. \tag{2.5}$$

The stability of the origin is proved in [[25], Sect. 34] when (2.1) or (2.3) hold and in [11] when (2.2) holds. These works assume that F is very smooth because their proofs are based on the Theorems of the invariant curve in [15, 25]. This requirement can be relaxed using more recent versions of this Theorem as in [6]. The instability of the origin is proved in [[25], Sect. 31] and in [10] when (2.5) or (2.4) holds.

The next definition will be essential for the rest of the paper.

DEFINITION 2.3. *Let $F \in A^3$ be given. The fixed point $z = 0$ is said to be of twist type if it is elliptic and* (2.1) *or* (2.2) *holds.*

According to Proposition 2.2, a fixed point of twist type is stable when F is sufficiently smooth. In such case there exist invariant curves around the origin and there is a change of variables that transforms the region between two invariant curves in an annulus in such a way that the Theorems of Birkhoff and Mather on existence of periodic and quasi-periodic points [9] can be applied.

It is interesting to notice that in case (2.3) the stability is decided by the third order terms and however the fixed point is not considered of twist type. Next we present a computable formula for β and γ.

PROPOSITION 2.4. *In the conditions of Proposition* 2.1 *assume that F has a Taylor expansion of the form*

$$F(z, \bar{z}) = \lambda z + F_2(z, \bar{z}) + F_3(z, \bar{z}) + \cdots$$

with

$$\lambda = e^{-i\theta}, \qquad F_2(z, \bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2,$$

$$F_3(z, \bar{z}) = Mz^3 + Nz^2\bar{z} + Pz\bar{z}^2 + Q\bar{z}^3.$$

Then the following formulas hold:
If $\lambda^n \neq 1$, $n = 1, 2, 3, 4$

$$\beta = \Im(\bar{\lambda} N) + \frac{3 \sin \theta}{1 - \cos \theta} |A|^2 + \frac{\sin 3\theta}{1 - \cos 3\theta} |C|^2. \tag{2.6}$$

If $\lambda = \pm i$,

$$\beta = \mp \Im(iN) \mp 3\, |A|^2 \pm |C|^2, \qquad |\gamma| = |Q - 2\bar{A}C|. \tag{2.7}$$

The formula (2.6) is stated in [3] and a proof can be seen in [20]. The formulas in (2.7) are obtained by similar computations.

To end this Section we show that fixed points of twist type are preserved by changes of variables that are symplectic transformations with multiplier in the sense of [12].

LEMMA 2.5. *Let $F, G \in A^3$ be given and assume that in some neighborhood of the origin $U \subset \mathbb{C}$ there exists $H \in C^3(U, \mathbb{C})$ such that*

$$F \circ H = H \circ G$$

and

$$|\partial_z H|^2 - |\partial_{\bar{z}} H|^2 = \lambda,$$

for some constant $\lambda > 0$. Then, if $z = 0$ is of twist type with respect to F, the same holds with respect to G.

Proof. We decompose H in the form $H = D_{\sqrt{\lambda}} \circ K$ with $K \in A^3$ and $D_{\sqrt{\lambda}} = \sqrt{\lambda}\, z$. Then, since β and $|\gamma|$ are symplectic invariants,

$$\beta(F) = \lambda \beta(D_{\sqrt{\lambda}}^{-1} \circ F \circ D_{\sqrt{\lambda}}) = \lambda \beta(G), \qquad |\gamma(F)| = \lambda\, |\gamma(G)|.$$

3. TWIST PERIODIC SOLUTIONS

Consider the Newton's equation

$$x'' = f(t, x) \tag{3.1}$$

where $t \in \mathbb{R}$, $x \in I$, $I \subset \mathbb{R}$ is an open interval, $f \in C^{0,3}(\mathbb{R}/T\mathbb{Z} \times I)$. Given $t_0 \in \mathbb{R}$, $z = q + ip \in \mathbb{C}$ with $q \in I$, the solution of (3.1) with initial conditions

$$x(t_0) = q, \qquad x'(t_0) = p$$

is denoted by $\Xi(t; t_0, z, \bar{z})$. The Theorem of Liouville implies that $z \to \Xi(t; t_0, z, \bar{z})$ is area-preserving.

The Poincaré map with starting time t_0 is defined by

$$P_{t_0}\colon \Omega_{t_0} \subset \mathbb{C} \to \mathbb{C}, \qquad P_{t_0}(z, \bar{z}) = \Xi(t_0 + T; t_0, z, \bar{z}) + i\Xi'(t_0 + T; t_0, z, \bar{z}),$$

$$\Omega_{t_0} = \{z \in \mathbb{C};\ \Xi(t; t_0, z, \bar{z}) \text{ exists in } [t_0, t_0 + T]\}.$$

Let φ be a T-periodic solution of (3.1) satisfying $z_0 = \varphi(t_0) + i\varphi'(t_0)$. Then z_0 is a fixed point of P_{t_0} and the mapping $F_{t_0} = \Sigma_{z_0}^{-1} \circ P_{t_0} \circ \Sigma_{z_0}$, with $\Sigma_{z_0}(z, \bar{z}) = z + z_0$ is in the class $A^3(D)$ for some disk D.

DEFINITION 3.1. *The solution φ is said to be of twist type if $z=0$ is of twist type as a fixed point of F_{t_0}.*

Remarks. 1. The previous concept is independent of the choice of $t_0 \in \mathbb{R}$. Given $t_0, t_0^* \in \mathbb{R}$, the periodicity in time of (3.1) implies that (in some neighborhood of the origin)

$$P_{t_0} \circ H = H \circ P_{t_0^*}$$

with $H(z, \bar{z}) = \Xi(t_0; t_0^*, z, \bar{z}) + i\Xi'(t_0; t_0^*, z, \bar{z})$. Since $\Sigma_{z_0}^{-1} \circ H \circ \Sigma_{z*} \in A^3$, the coefficients β and $|\gamma|$ of the normal forms of F_{t_0} and $F_{t_0^*}$ coincide.

2. It is also interesting to remark that a twist periodic solution remains so after a translation and a change of scale of the independent variable. In fact, after the change

$$\xi = x, \qquad \tau = \frac{t - t_0}{\alpha^2} \qquad (\alpha^2 > 0)$$

the equation becomes

$$\frac{d^2\xi}{d\tau^2} = \alpha^4 f(\alpha^2 \tau + t_0, \xi)$$

and the periodic solution is now $\varphi^*(\tau) = \varphi(\alpha^2 \tau + t_0)$ with period $T^* = T/\alpha^2$. The Poincaré maps of both equations are conjugate in the sense of Lemma 2.5 and therefore φ^* is also of twist type.

3. The definition of solution of twist type given in [20] differs slightly from the previous one. In [20] it was assumed that z_0 was an elliptic fixed point satisfying (2.1) and the alternative (2.2) was excluded.

4. It follows from the comments after Definition 2.3 that if φ is a solution of twist type and $f \in C^{0,5}$ then φ is Lyapunov stable. In addition, there exist subharmonic and quasi-periodic solutions accumulating at φ (see [[25], Sect. 36]).

For simplicity we assume that $\varphi = 0$ and (3.1) can be rewritten in the form

$$x'' + a(t)\, x + b(t)\, x^2 + c(t)\, x^3 + r(t, x) = 0, \tag{3.2}$$

where $a, b, c \in C(\mathbb{R}/T\mathbb{Z})$, $r \in C^{0,3}(\mathbb{R}/T\mathbb{Z} \times (-\varepsilon, \varepsilon))$ and $\partial^\alpha r(t, 0) = 0$, $\forall t \in \mathbb{R}$, $\alpha = 0, 1, 2, 3$. (Of course this is always the case after the change of variables $y = x - \varphi(t)$).

The linearization of (3.2) at $x = 0$ is the Hill's equation

$$y'' + a(t)\, y = 0 \tag{3.3}$$

and we shall assume that

$$\text{the distance between two consecutive zeros of a nontrivial solution of (3.3) is at least } T. \tag{3.4}$$

With respect to the nonlinear terms we assume

$$\begin{gathered} c(t) \leqslant 0 \qquad \forall t \in \mathbb{R}, \\ b(t) \leqslant 0 \qquad \forall t \in \mathbb{R} \qquad \text{or} \qquad b(t) \geqslant 0 \qquad \forall t \in \mathbb{R}, \\ \int_0^T |b| + |c| > 0. \end{gathered} \tag{3.5}$$

THEOREM 3.2. *There exists a number θ^*, $\pi/2 < \theta^* < 2\pi/3$, such that if* (3.3) *has Floquet multipliers $e^{\pm i\theta}$ with*

$$0 < |\theta| \leqslant \theta^* \qquad \text{or} \qquad \frac{2\pi}{3} < |\theta| < \pi \tag{3.6}$$

and (3.4), (3.5) *hold, then $x = 0$ is of twist type.*

The proof of the Theorem will be postponed to Section 4 where it is shown that an admissible value of θ^* is 1.6507.... The optimal value of θ^* is not known to the author. Let us denote it by θ^*_{op}. An example at the end of this Section will provide an upper bound of θ^*_{op}. In particular, $\theta^*_{op} < 2\pi/3$.

Figure 1 illustrates the regions of the unit circle where the multipliers lie when (3.6) holds ($\xi = e^{i\theta^*}$).

To verify the assumptions (3.4), (3.6) one can use the techniques that are tradionally employed in the study of Hill's equation (see [7]). We present

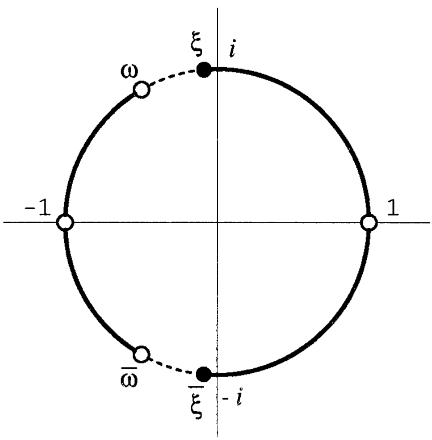

FIGURE 1

a corollary that is reminiscent of the classical criteria of Lyapunov and Zukovskii for stability of Hill's equation [26].

COROLLARY 3.3. *The trivial solution of* (3.2) *is of twist type when* (3.5) *and one of the following conditions are satisfied*:

(i) $a \leqslant (\theta^*/T)^2$ *and* (3.3) *is elliptic*

(ii) $0 \ll a \leqslant (\theta^*/T)^2$ *or* $(2\pi/3T)^2 \ll a \ll (\pi/T)^2$

(iii) $T \int_0^T a^+(t)\,dt \leqslant 4(\theta^*/\pi)^2$ *and* (3.3) *is elliptic*

(iv) $a \geqslant 0,\ T \int_0^T a^+(t)\,dt \leqslant 4(\theta^*/\pi)^2$.

(*The notation* $f \ll g$ *means* $f \leqslant g$ *with strict inequality on a set of positive measure*, $a^+ = \max\{a, 0\}$).

Remarks. 1. The case (i) extends the main result in [20], where it was assumed that

$$a \leqslant \left(\frac{\pi}{3T}\right)^2.$$

2. When $b = 0$ the trivial solution $x = 0$ is of twist type as soon as it is elliptic and $c \geqslant 0$ or $c \ll 0$. This follows from [21].

To prove this corollary it will be sufficient to verify the conditions (3.4), (3.6) and apply Theorem 3.2. To do this we shall use the geometric presentation of Floquet theory given in [17]. Let us review it.

Let y be a nontrivial solution of (3.3) and introduce polar coordinates

$$y = r \sin\theta, \qquad y' = r\cos\theta, \qquad r > 0.$$

The angle θ satisfies

$$\theta' = \cos^2\theta + a(t)\sin^2\theta. \tag{3.7}$$

The theory of differential equations on a torus (see for instance [24]) implies that the limit

$$\alpha := \lim_{t \to +\infty} \frac{\theta(t)}{t}$$

exists and is independent of the initial condition $\theta(0)$. We refer to α as the rotation number of (3.3) and write $\alpha = \alpha(a)$. The rotation number has the following properties, that are proved in [17],

(1) $\alpha \geqslant 0$

(2) $a_1 \leqslant a_2$ [resp. $a_1 \lll a_2$] $\Rightarrow \alpha(a_1) \leqslant \alpha(a_2)$ [resp. $\alpha(a_1) < \alpha(a_2)$]

(3) If (3.3) is elliptic, the Floquet multipliers are $e^{\pm iT\alpha}$.

Before the proof of the corollary we need a preliminary result on (3.7).

LEMMA 3.4. *Let $\theta(t)$ be a solution of* (3.7) *such that for some $t_0 < t_1$, $\theta(t_0) = p\pi$, $\theta(t_1) = (p+m)\pi$ with $p \in \mathbb{Z}$, $m \in \mathbb{N}$. Then*

$$(t_1 - t_0) \int_{t_0}^{t_1} a^+(t)\, dt \geqslant 4m^2.$$

Proof. For simplicity we prove the result for $m = 1$, $p = 0$. Given $\lambda > 0$ let f: $\mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \arctan(\lambda \tan x) \qquad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

$$f(x+\pi) = f(x) + \pi \qquad \forall x \in \mathbb{R}.$$

Then f is a diffeomorphism and the change of variables $v(t) = f(\theta(t))$ transforms (3.7) into

$$v' = \lambda \cos^2 v + \frac{a(t)}{\lambda} \sin^2 v.$$

Since $v(t_0) = 0$, $v'(t_0) > 0$, $v(t_1) = \pi$, $v'(t_1) > 0$ one can find $t_0 < \tau_0 \leqslant \tau_0^* < \tau_1 \leqslant \tau_1^* < t_1$ such that

$$0 < v(t) < \frac{\pi}{4}, \qquad t \in (t_0, \tau_0), \qquad v(\tau_0) = v(\tau_0^*) = \frac{\pi}{4},$$

$$\frac{\pi}{4} < v(t) < \frac{3\pi}{4}, \qquad t \in (\tau_0^*, \tau_1), \qquad v(\tau_1) = v(\tau_1^*) = \frac{3\pi}{4},$$

$$\frac{3\pi}{4} < v(t) < \pi, \qquad t \in (\tau_1^*, t_1).$$

In (t_0, τ_0), $|\sin v| \leqslant |\cos v|$ and v satisfies the differential inequality

$$v' \leqslant \left(\lambda + \frac{a^+(t)}{\lambda} \right) \cos^2 v.$$

Integrating,

$$1 = \int_0^{\pi/4} \frac{dv}{\cos^2 v} \leqslant \int_{t_0}^{\tau_0} \left\{ \lambda + \frac{a^+(t)}{\lambda} \right\} dt.$$

In an analogous way one obtains

$$2 \leqslant \int_{\tau_0^*}^{\tau_1} \left\{ \lambda + \frac{a^+(t)}{\lambda} \right\} dt, \qquad 1 \leqslant \int_{\tau_1^*}^{t_1} \left\{ \lambda + \frac{a^+(t)}{\lambda} \right\} dt.$$

In consequence $\int_{t_0}^{t_1} \{\lambda + (a^+(t)/\lambda)\}\, dt \geqslant 4$ and the conclusion is reached letting $\lambda = \sqrt{(1/t_1 - t_0) \int_{t_0}^{t_1} a^+}$.

Proof of Corollary 3.3. (i) It follows from Sturm comparison theory that (3.4) holds as soon as $a \leqslant (\pi/T)^2$. To check (3.6) we first remark that the rotation number of the equation of constant coefficients $y'' + \omega^2 y = 0$, with $\omega > 0$, is $\alpha(\omega^2) = \omega$. Now, by comparison of this equation with (3.3), $\alpha(a) \leqslant \theta^*/T$ and property 3) of the rotation number implies that the multipliers of (3.3) are $e^{\pm i\alpha(a)\, T}$ with $0 < \alpha(a)\, T \leqslant \theta^*$.

(ii) The proof is similar to (i).

(iii) Let y be a nontrivial solution of (3.3). Two consecutive zeros of y, say t_0, t_1, are such that $\theta(t_0) = p\pi$, $\theta(t_1) = (p+1)\pi$. In consequence, the previous Lemma implies that (3.4) holds as soon as $T \int_0^T a^+ \leqslant 4$. To check (3.6) let $\theta(t)$ be a solution of (3.7). We can deduce (3.6) from the following property

$$T \int_0^T a^+ \leqslant 4\sigma^2 \qquad \Rightarrow \qquad \alpha \leqslant \sigma \frac{\pi}{T}.$$

To prove this last statement it is sufficient to consider the case of $\sigma \in \mathbb{Q}$, say $\sigma = m/n$. Then, for each $k \geqslant 1$, $knT \int_0^{knT} a^+ \leqslant 4km^2$ and, from Lemma 3.4, $\theta(knT) \leqslant km\pi$, where $\theta(t)$ is the solution of (3.7) with $\theta(0) = 0$. The definition of α shows that $\alpha \leqslant (m/n)(\pi/T)$.

(iv) The proof of Lyapunov criterion for Hill's equation shows that in this case (3.3) is elliptic and therefore (iii) applies.

We shall finish this Section with two examples. The first example shows that the sign of c cannot be changed in (3.5). The second will prove that if the multipliers do not lie in the region indicated by (3.6), the trivial solution is not always of twist type.

EXAMPLE 3.4. The equation

$$y'' + y - \frac{1}{y^3} = 0, \qquad y > 0$$

is a particular case of the equation integrated in [23] and the solution is

$$y(t; q, p) = \sqrt{q^2 \cos^2 t + \left(p^2 + \frac{1}{q^2}\right) \sin^2 t + 2pq \cos t \sin t}, \qquad q > 0, \quad p \in \mathbb{R}.$$

For $q = 1$, $p = 0$ one obtains the equilibrium $y = 1$ and for the rest of initial conditions the solution is periodic with period π. After the change $x = y - 1$ the equation is expressed in the form (3.2) with

$$x'' + 4x - 6x^2 + 10x^3 + \cdots = 0.$$

If we look at this autonomous equation as a periodic equation with arbitrary period $T > 0$, the trivial solution cannot be of twist type. In fact, when $T \notin \pi\mathbb{Q}$, the only nT-periodic solution ($n \geqslant 1$) is $x = 0$ and this would violate Birkhoff Theorem if $x = 0$ were of twist type. When $T \in \pi\mathbb{Q}$ there exists $n \in \mathbb{N}$ such that $P_{t0}{}^n = I$ and therefore $x = 0$ is not of twist type.

On the other hand we notice that $b = -6 < 0$, $c = 10 > 0$ and (3.4) is satisfied if $T \leqslant \pi/2$. The multipliers of $x'' + 4x = 0$ are $e^{\pm 2iT}$. In consequence, if $T \leqslant \theta/2$ or $\pi/3 < T < \pi/2$ also (3.6) holds. Thus, in these conditions the trivial solution is not of twist type and the assumptions of the Theorem hold excepting that c is positive instead of negative.

EXAMPLE 3.5. We look at the period T as a parameter with $T \geqslant \pi/2$ and consider the equation

$$x'' + x + b_T(t)\, x^2 = 0$$

where b_T is T-periodic and

$$b_T(t) = \begin{cases} \sin 2t, & \text{if} \quad t \in \left[0, \dfrac{\pi}{2}\right] \\[2ex] 0, & \text{if} \quad t \in \left(\dfrac{\pi}{2}, T\right). \end{cases}$$

The linearized equation is $x'' + x = 0$ and has the multipliers $e^{\pm iT}$. The condition (3.4) is verified when $T \leqslant \pi$ and (3.5) is always valid. In consequence, Theorem 3.2 can be applied whenever

$$\frac{\pi}{2} \leqslant T \leqslant \theta^* \qquad \text{or} \qquad \frac{2\pi}{3} < T < \pi.$$

Next we apply the instability result in [20]. When $T = 2\pi/3$ the trivial solution is unstable because $\int_0^T b_T(t)\, e^{3it}\, dt \neq 0$. This is not surprising since we are in the presence of a strong resonance at the third root of unity. We now prove the existence of $T^* \in (\theta^*, 2\pi/3)$ such that $x = 0$ is not of twist type for $T = T^*$. This example shows that (3.6) is indeed needed in the statement of the Theorem.

The Birkhoff coefficient β in the normal form of the Poincaré map can be thought as a function of T, $\beta = \beta(T)$. This is related to the approach taken in [14]. A direct computation shows that

$$\beta(T) = \frac{352 - 30\pi}{2400} + \frac{1}{6}\frac{\sin T}{1 - \cos T} + \frac{1}{50}\frac{\sin 3T}{1 - \cos 3T}. \tag{3.8}$$

(A proof of this formula will be obtained in Section 4, after Proposition 4.4). It is easy to prove that the equation $\beta(T) = 0$ has a unique root T^* in the interval $(\pi/2, 2\pi/3)$. At $T = T^*$ the trivial solution is not of twist type.

4. Proof of the Theorem

4.1. *The Linearized Equation*

Let us consider the Hill's equation

$$y'' + a(t)\, y = 0, \qquad a \in C(\mathbb{R}/T\mathbb{Z}) \tag{4.1}$$

and let $\Psi = \phi_1 + i\phi_2$ be the solution of (4.1) satisfying

$$\Psi(0) = 1, \qquad \Psi'(0) = i.$$

The solution Ψ is complex-valued and ϕ_1, ϕ_2 are respectively the real and imaginary parts of Ψ.

Sometimes it will be convenient to write Ψ in polar coordinates

$$\Psi(t) = r(t)\, e^{i\varphi(t)}, \qquad t \in \mathbb{R}$$

where $r, \varphi \in C^2(\mathbb{R})$ and $\varphi(0) = 0$. This is always possible since $\Psi(t)$ never vanishes. A computation shows that

$$\varphi' = \frac{\phi_1\phi_2' - \phi_2\phi_1'}{r^2} = \frac{1}{r^2} > 0$$

and therefore φ is increasing.

The equation (4.1) will be called *R*-elliptic is there exists $\lambda \in S^1$, $\lambda \neq \pm 1$ such that

$$\Psi(t + T) = \bar{\lambda}\Psi(t) \qquad \forall t \in \mathbb{R}.$$

The condition of R-ellipticity is equivalent to saying that the monodromy matrix

$$\begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \phi_1'(T) & \phi_2'(T) \end{pmatrix}$$

is a rotation (different from $\pm I$). In which case λ and $\bar{\lambda}$ are the characteristic multipliers of (4.1).

It follows from Proposition 7 in [20] that every elliptic equation of the kind (4.1) can be transformed into a R-elliptic equation by scaling and translating the time variable. We state this fact in precise terms.

LEMMA 4.1. *Assume that* (4.1) *is elliptic. Then there exists* t_0 *and* $\alpha > 0$ *such that the change of variables* $z = y$, $\tau = (t - t_0)/\alpha^2$ *transforms* (4.1) *in*

$$\frac{d^2 z}{d\tau^2} + a^*(\tau)\, z = 0, \quad a^* \in C(\mathbb{R}/T^*\mathbb{Z}), \qquad T^* = \frac{T}{\alpha^2}$$

and this equation is R-elliptic.

When (4.1) is R-elliptic the polar form of Ψ has some special properties; namely r is T-periodic and φ satisfies

$$\varphi(t+T) = \theta + \varphi(t) + 2n\pi, \qquad \forall t \in \mathbb{R}$$

with $\theta \in \operatorname{Arg}(\bar{\lambda})$ and $n \in \mathbb{Z}$. This property can be sharpened when the disconjugacy condition (3.4) is satisfied.

LEMMA 4.2. *Assume that* (4.1) *is R-elliptic and* (3.4) *holds. Then*

$$\varphi(t+T) = \theta + \varphi(t), \qquad \forall t \in \mathbb{R}$$

with $0 < \theta < \pi$.

Proof. The condition (3.4) implies that the solution ϕ_2 is positive on $(0, T)$. In consequence $\Psi(t)$ lies on the half plane $\Im z > 0$ if $t \in (0, T)$. Since φ is increasing and $\varphi(0) = 0$, the angle $\theta = \varphi(T)$ belongs to the interval $(0, \pi]$. The extreme point π is now excluded because the equation is elliptic.

4.2. *The Taylor Expansion of the Poincaré Map*

Let $P(z, \bar{z})$ be the Poincaré map of (3.2) and assume that (3.3) is R-elliptic. Then P has the expansion

$$P(z, \bar{z}) = \lambda z + P_2(z, \bar{z}) + P_3(z, \bar{z}) + \cdots$$

where λ is given by the definition of R-ellipticity and P_2, P_3 are homogeneous polynomials of degree 2 and 3 respectively. We wish to compute P_2, P_3 and this will be done using the same approach of [21].

Let $x(t; z, \bar{z})$ be the solution of (3.2) with

$$x(0) = q, \qquad x'(0) = p, \qquad z = q + ip.$$

The theorem of differentiability with respect to initial conditions implies that

$$x(t; z, \bar{z}) = \tfrac{1}{2}\,[\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}] + O(|z|^2), \qquad z \to 0, \tag{4.2}$$

and this expansion is uniform with respect to $t \in [0, T]$.

To compute the higher order derivatives we need a simple result on the linear non-homogeneous equation

$$y'' + a(t)\, y + f(t) = 0, \qquad f \in C[0, T]. \tag{4.3}$$

LEMMA 4.3. *Let y be the solution of* (4.3) *with $y(0) = y'(0) = 0$. Then*

$$y(t) = \int_0^t G(t, s)\, f(s)\, ds, \qquad t \in [0, T]$$

with $G(t, s) := \phi_1(t)\, \phi_2(s) - \phi_2(t)\, \phi_1(s)$. Moreover, if (4.1) *is R-elliptic,*

$$y(T) + iy'(T) = -i\lambda \int_0^T f(t)\, \Psi(t)\, dt.$$

We look at the nonlinear equation (3.2) as an equation of the kind (4.3) with

$$f(t) = b(t)\, x(t)^2 + c(t)\, x(t)^3 + r(t, x(t)). \tag{4.4}$$

Then

$$x(t; z, \bar{z}) = \tfrac{1}{2}\,[\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}] + \int_0^t G(t, s)\, f(s)\, ds$$

and, combining this expression with (4.2), one gets (uniformly in $t \in [0, T]$)

$$x(t; z, \bar{z}) = \frac{1}{2}\,[\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}] + \int_0^t G(t, s)\, b(s) \left(\frac{\bar{\Psi}(s)\, z + \Psi(s)\, \bar{z}}{2}\right)^2 ds$$
$$+ O(|z|^3), \qquad z \to 0. \tag{4.5}$$

Since we assumed that (3.3) was R-elliptic we can also apply the second part of Lemma 4.3 to (3.2) to obtain

$$P(z, \bar{z}) = \lambda z - i\lambda \int_0^T f(t)\, \Psi(t)\, dt, \tag{4.6}$$

with f given by (4.4).

Combining (4.6) and (4.5) we are lead to the identities

$$P_2(z, \bar{z}) = -i\lambda \int_0^T b(t) \left(\frac{\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}}{2} \right)^2 \Psi(t)\, dt,$$

$$\begin{aligned} &P_3(z, \bar{z}) \\ &= -i\lambda \int_0^T c(t) \left(\frac{\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}}{2} \right)^3 \Psi(t)\, dt \\ &\quad - 2i\lambda \int_0^T b(t) \left(\frac{\bar{\Psi}(t) z + \Psi(t) \bar{z}}{2} \right) \left\{ \int_0^t G(t, s) b(s) \left(\frac{\bar{\Psi}(s) z + \Psi(s) \bar{z}}{2} \right)^2 ds \right\} \Psi(t)\, dt. \end{aligned}$$

To simplify P_3 we use Fubini's theorem on the triangle $\Delta_T = \{(t, s) \in \mathbb{R}^2;\ 0 < s < t,\ 0 < t < T\}$ and obtain

$$\begin{aligned} &P_3(z, \bar{z}) \\ &\quad = -i\lambda \int_0^T c(t) \left(\frac{\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}}{2} \right)^3 \Psi(t)\, dt \\ &\qquad - i\lambda \iint_{\Delta_T} G(t, s)\, b(t)\, b(s) (\bar{\Psi}(t)\, z + \Psi(t)\, \bar{z}) \left(\frac{\bar{\Psi}(s)\, z + \Psi(s)\, \bar{z}}{2} \right)^2 \Psi(t)\, ds\, dt. \end{aligned}$$

We use the same notations of Section 2 and specify the coefficients involved in the computation of the normal form.

PROPOSITION 4.4. *Assume that* (3.3) *is R-elliptic and*

$$P_2(z, \bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2, \qquad P_3(z, \bar{z}) = Mz^3 + Nz^2\bar{z} + Pz\bar{z}^2 + Q\bar{z}^3.$$

Then

$$A = -\frac{i\lambda}{4} \int_0^T b(t)\, \bar{\Psi}(t)^2\, \Psi(t)\, dt, \qquad C = -\frac{i\lambda}{4} \int_0^T b(t)\, \Psi(t)^3\, dt$$

$$\begin{aligned} N = &-\frac{3i\lambda}{8} \int_0^T c(t)\, |\Psi(t)|^4\, dt \\ &-\frac{i\lambda}{4} \iint_{\Delta_T} G(t, s)\, b(t)\, b(s) [2\, |\Psi(t)|^2\, |\Psi(s)|^2 + \Psi(t)^2\, \bar{\Psi}(s)^2]\, ds\, dt \end{aligned}$$

$$Q = -\frac{i\lambda}{8} \int_0^T c(t)\, \Psi(t)^4\, dt - \frac{i\lambda}{4} \iint_{\Delta_T} G(t, s)\, b(t)\, b(s)\, \Psi(t)^2\, \Psi(s)^2\, ds\, dt.$$

CONTINUATION OF EXAMPLE 3.5. Since $T\in(\pi/2, 2\pi/3)$, the equation $x''+x=0$ is R-elliptic with $\Psi(t)=e^{it}$. In the previous notations,

$$A=-\frac{ie^{-iT}}{4}\int_0^{\pi/2}\sin 2te^{-it}\,dt=-\frac{ie^{-iT}}{4}\left(\frac{2}{3}-\frac{2}{3}i\right)$$

$$B=-\frac{ie^{-iT}}{4}\int_0^{\pi/2}\sin 2te^{3it}\,dt=-\frac{ie^{-iT}}{4}\left(-\frac{2}{5}+\frac{2}{5}i\right)$$

$$\Im(\bar{\lambda}N)=\frac{1}{4}\int_0^{\pi/2}dt\int_0^t ds\,\sin(t-s)\sin 2t\sin 2s[2+\cos 2(t-s)]$$

$$=\frac{1}{4}\frac{352-30\pi}{600}.$$

Now (3.8) follows from (2.6).

4.3 *An Integral Inequality*

We now obtain an inequality that concerns the functions in the pointed cone

$$L_+=\left\{f\in L^1(I);\, f\geqslant 0 \text{ a.e.}, \int_I f>0\right\}.$$

where $I\subset\mathbb{R}$ is a bounded interval.

LEMMA 4.5. *Assume that $|I|\leqslant\pi/2$. Then*

$$\left|\int_I f(t)\,e^{3it}\,dt\right|<\left|\int_I f(t)\,e^{it}\,dt\right| \qquad \forall f\in L_+.$$

Remark. The previous inequality has an extension to positive measures. Let $M(I)$ be the space on measures on I and

$$M_+=\left\{\mu\in M(I);\, \mu\geqslant 0, \int_I d\mu>0\right\}.$$

The previous lemma and an approximation argument imply

$$\left|\int_I e^{3it}\,d\mu(t)\right|\leqslant\left|\int_I e^{it}\,d\mu(t)\right| \qquad \forall\mu\in M_+.$$

This inequality is not strict when $\mu=\delta_{t_0}$ (the Dirac measure concentrated at $t=t_0$). This proves that the inequality in Lemma 4.5 is optimal.

Proof. It is not restrictive to assume $I=(0,\theta)$, $\theta\leqslant\pi/2$. We first obtain the identity

$$\left|\int_I f(t)\,e^{nit}\,dt\right|^2=2\iint_\Delta f(t)\,f(s)\cos(n[t-s])\,ds\,dt, \qquad n=1,3. \quad (4.7)$$

Here, $\Delta=\{(t,s)\in\mathbb{R}^2;\ 0<s<t,\ 0<t<\theta\}$. The lemma now follows from (4.7) and the trigonometric inequality

$$\cos 3t<\cos t \qquad \forall t\in\left(0,\frac{\pi}{2}\right).$$

To prove (4.7) one uses Fubini's Theorem,

$$\left|\int_I f(t)\,e^{nit}\,dt\right|^2=\left(\int_I f(t)\,e^{nit}\,dt\right)\left(\int_I f(s)\,e^{-nis}\,ds\right)$$

$$=\iint_{I\times I} f(t)\,f(s)\,e^{ni(t-s)}\,ds\,dt.$$

Now $\iint_{I\times I}=\iint_\Delta+\iint_{\Delta^*}$, where $\Delta^*=\{(t,s)\in\mathbb{R}^2;\ 0<t<s,\ 0<s<\theta\}$. The change of variables $(t,s)\to(s,t)$ from Δ^* onto Δ ends the proof.

LEMMA 4.6. *Assume that* $|I|<\pi$. *Then*

$$\left|\int_I f(t)\,e^{3it}\,dt\right|<\frac{1}{\cos(|I|/2)}\left|\int_I f(t)\,e^{it}\,dt\right| \qquad \forall f\in L_+.$$

Remarks. 1. This inequality is not optimal in general. In the special case $|I|=2\pi/3$ it becomes

$$\left|\int_I f(t)\,e^{3it}\,dt\right|<2\left|\int_I f(t)\,e^{it}\,dt\right| \qquad \forall f\in L_+$$

and it is optimal. Actually, if one assumes $I=(0,2\pi/3)$, the equality is reached for the measure $\mu=\delta_0+\delta_{2\pi/3}$. I thank J. Campos and J. L. Vázquez for pointing out this fact.

2. It can be proved that an inequality of the kind

$$\left|\int_I f(t)\,e^{3it}\,dt\right|<k\left|\int_I f(t)\,e^{it}\,dt\right| \qquad \forall f\in L_+$$

holds if and only if $|I|\leqslant\pi$.

Proof. Assume $I=(0,\theta)$, $\theta<\pi$. It follows from Cauchy–Schwarz inequality that

$$\left|\int_I f(t)\,e^{nit}\,dt\right| = \max\left\{\int_I f(t)\cos(nt+\phi)\,dt;\ \phi\in\mathbb{R}\right\}, \qquad n=1,3.$$

We shall combine this fact with the trigonometric inequality

$$\cos\left(t-\frac{\theta}{2}\right) > \cos\frac{\theta}{2}, \qquad \forall t\in(0,\theta).$$

For each $\phi\in\mathbb{R}$,

$$\int_I f(t)\cos(3t+\phi)\,dt \leqslant \int_I f(t)\,dt < \frac{1}{\cos\theta/2}\int_I f(t)\cos\left(t-\frac{\theta}{2}\right)dt$$
$$\leqslant \frac{1}{\cos\theta/2}\left|\int_I f(t)\,e^{it}\,dt\right|.$$

4.4. *Proof of the Theorem*

The change of variables given in Lemma 4.1 transforms the equation (3.2) in another equation of the same kind that also satisfies the assumptions of the theorem. In view of remark 2 after definition 3.1, it is not restrictive to assume that the change has been already performed and (3.3) is R-elliptic. It follows from Lemma 4.2 that the angle φ verifies

$$\varphi(t+T)=\theta+\varphi(t), \qquad t\in\mathbb{R}$$

with $\lambda=e^{-i\theta}$, $\theta\in(0,\pi)$. Before proceeding to prove the theorem we obtain estimates on the second and third order derivatives of P.

Lemma 4.7. *In the notations of Proposition* 4.4,

$$\text{(i)}\quad |C|\leqslant|A| \quad \textit{if}\quad \theta\leqslant\frac{\pi}{2}, \qquad \text{(ii)}\quad |C|\leqslant\frac{1}{\cos\theta/2}\,|A| \quad \textit{if}\quad \theta\geqslant\frac{\pi}{2}.$$

Moreover, the equality only holds when $A=C=0$.

Proof. We express the coefficients A and C in terms of the polar form of Ψ,

$$A=-\frac{i\lambda}{4}\int_0^T b(t)\,r(t)^3\,e^{-i\varphi(t)}\,dt, \qquad C=-\frac{i\lambda}{4}\int_0^T b(t)\,r(t)^3\,e^{3i\varphi(t)}\,dt.$$

The change of variables $s=\varphi(t)$ leads to,

$$A=-\frac{i\lambda}{4}\int_0^\theta f(s)\,e^{-is}\,ds, \qquad C=-\frac{i\lambda}{4}\int_0^\theta f(s)\,e^{3is}\,ds,$$

where $f(s)=b(\varphi^{-1}(s))\,r(\varphi^{-1}(s))^3\,1/\varphi'(\varphi^{-1}(s))$. Since $\varphi'>0$ and (3.5) holds, we can apply Lemmas 4.5 and 4.6 to end the proof.

LEMMA 4.8. *In the notations of Proposition* 4.4

$$\Im(\bar{\lambda}N)>|Q|.$$

Proof. First we remark that the assumption (3.4) implies that the function $G(t,s)$ is strictly negative whenever $0<s<t<T$. (Since $\phi_2>0$ on $(0,T)$, $(\phi_1/\phi_2)'=\phi_1'\phi_2-\phi_1\phi_2'/\phi_2^2=-1/\phi_2^2<0$ and ϕ_1/ϕ_2 is decreasing or, equivalently, G is negative). From Proposition 4.4 and (3.5),

$$\Im(\bar{\lambda}N)=-\tfrac{3}{8}\int_0^T c(t)\,|\Psi(t)|^4\,dt$$
$$-\tfrac{1}{4}\iint_{\Delta_T} G(t,s)\,b(t)\,b(s)\,[\,2\,|\Psi(t)|^2\,|\Psi(s)|^2+\Re\{\Psi(t)^2\,\bar{\Psi}(s)^2\}\,]\,ds\,dt,$$
$$|Q|\leqslant-\tfrac{1}{8}\int_0^T c(t)\,|\Psi(t)|^4\,dt-\tfrac{1}{4}\iint_{\Delta_T} G(t,s)\,b(t)\,b(s)\,|\Psi(t)|^2\,|\Psi(s)|^2\,ds\,dt.$$

Define $\Gamma=\{(t,s)\in\Delta_T;\ \varphi(t)-\varphi(s)=\pi/2\}$. The following inequality holds for every $(t,s)\in\Delta_T-\Gamma$,

$$2\,|\Psi(t)|^2\,|\Psi(s)|^2+\Re\{\Psi(t)^2\,\bar{\Psi}(\mathrm{s})^2\}>|\Psi(t)|^2\,|\Psi(s)|^2.$$

The set Γ is a smooth curve and, therefore, it has measure zero in $\mathbb{R}^2$. Thus, the previous inequality holds almost everywhere and the conclusion follows.

After these lemmas we prove the Theorem.

Define

$$\Phi(\theta):=\frac{3\sin\theta}{1-\cos\theta}+\frac{\sin 3\theta}{(1-\cos 3\theta)\cos^2\left(\dfrac{\theta}{2}\right)}.$$

This function is continuous in $[\pi/2, 2\pi/3)$ and $\Phi(\pi/2)=1$, $\lim_{\theta\to(2\pi/3)^-}\Phi(\theta)=-\infty$. Let θ^* be the first positive zero of Φ in $(\pi/2, 2\pi/3)$, so that

$$\Phi(\theta^*)=0, \qquad \Phi(\theta)>0, \qquad \forall\theta\in\left[\frac{\pi}{2},\theta^*\right).$$

(It is easy to check that θ^* is the unique root of Φ in the interval $(\pi/2, 2\pi/3)$ and $\theta^* = 1.6507...$). We now apply Proposition 2.4 and Lemmas 4.7 and 4.8 in each of the following four cases.

(i) $\theta \in (\pi/2, \theta^*]$.

$$\beta > \frac{3 \sin \theta}{1 - \cos \theta} |A|^2 + \frac{\sin 3\theta}{1 - \cos 3\theta} |C|^2 \geqslant \Phi(\theta) |A|^2 \geqslant 0.$$

(ii) $\theta = \pi/2$

$$\beta > |Q| + 3 |A|^2 - |C|^2 \geqslant |Q| + |A|^2 + |C|^2 \geqslant |Q| + 2 |A| |C| \geqslant |\gamma|.$$

(iii) $\theta \in (\pi/3, \pi/2)$.

We use the trigonometric inequality

$$\frac{3 \sin \theta}{1 - \cos \theta} + \frac{\sin 3\theta}{1 - \cos 3\theta} \geqslant 2 \qquad \forall \theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right).$$

Then

$$\beta > \frac{3 \sin \theta}{1 - \cos \theta} |A|^2 + \frac{\sin 3\theta}{1 - \cos 3\theta} |C|^2 \geqslant 2 |A|^2 \geqslant 0.$$

(iv) $\theta \in (0, \pi/3] \cup (2\pi/3, \pi)$.

In this case,

$$\frac{3 \sin \theta}{1 - \cos \theta} > 0 \qquad \frac{\sin 3\theta}{1 - \cos 3\theta} > 0$$

and

$$\beta > \frac{3 \sin \theta}{1 - \cos \theta} |A|^2 + \frac{\sin 3\theta}{1 - \cos 3\theta} |C|^2 \geqslant 0.$$

5. The Quadratic Equation

Consider the equation

$$x'' + b(t) x^2 = p(t), \tag{5.1}$$

where $b, p \in C(\mathbb{R}/T\mathbb{Z})$ and

$$b(t) > 0 \qquad \forall t \in \mathbb{R}, \qquad \frac{1}{T} \int_0^T b(t) \, dt = 1.$$

The nonlinear term $b(t)\,x^2$ is coercive and the results on existence of T-periodic solutions of [5] can be applied. They are described in a simpler way using the parametric equation

$$x'' + b(t)\,x^2 = p^*(t) + s, \tag{5.2}$$

where $p^* \in C(\mathbb{R}/T\mathbb{Z})$ is a fixed function and $s \in \mathbb{R}$ is a parameter. It follows from [5] that there exists s_0 (depending on p^*) such that (5.2) has T-periodic solutions if and only if $s \geqslant s_0$. Moreover, if $s > s_0$ there exist at least two such solutions.

We shall obtain results on the exact number of T-periodic solutions, their stability properties and the existence of subharmonic and quasi-periodic solutions. First we introduce some definitions. Let x be a T-periodic solution of (5.1). It is said that x is nondegenerate if the variational equation

$$y'' + 2b(t)\,x(t)\,y = 0 \tag{5.3}$$

has not T-periodic solutions different from zero. It is said that x is elliptic, parabolic or hyperbolic when it is the case for the variational equation. The previous definitions can be phrased in terms of the Floquet multipliers, μ_1, μ_2, of (5.3):

$$\begin{aligned}
\mu_i \neq 1, \qquad i = 1, 2 &\Leftrightarrow x \text{ nondegenerate}\\
|\mu_1| = |\mu_2| = 1, \qquad \mu_i \neq \pm 1 &\Leftrightarrow x \text{ elliptic,}\\
\mu_1 = \mu_2 = \pm 1 &\Leftrightarrow x \text{ parabolic}\\
|\mu_1| < 1 < |\mu_2| &\Leftrightarrow x \text{ hyperbolic.}
\end{aligned}$$

A hyperbolic solution is always unstable. Also, it is clear that a solution of twist type is elliptic.

THEOREM 5.1. *Assume that*

$$T^3 \int_0^T p(t)\,dt \leqslant 4 \left(\frac{\theta^*}{\pi}\right)^4. \tag{5.4}$$

Then one of the alternatives below holds:

(o) *The equation* (5.1) *has no T-periodic solutions*

(i) *There exists a unique T-periodic solution of* (5.1) *that is parabolic and unstable*

(ii) *There exist exactly two T-periodic solutions of* (5.1), *one of them hyperbolic and another of twist type.*

Remarks. 1. Assume that b is fixed and let M_k, $k=0, 1, 2$, denote the class of functions $p \in C(\mathbb{R}/T\mathbb{Z})$ such that (5.4) and alternative (k) are satisfied. Using the techniques of [5] one can prove that if $p_1, p_2 \in C(\mathbb{R}/T\mathbb{Z})$ satisfy (5.4) and $p_1 \ll p_2$ then

$$p_1 \in M_1 \cup M_2 \Rightarrow p_2 \in M_2$$

$$p_2 \in M_0 \cup M_1 \Rightarrow p_1 \in M_0.$$

These implications can be employed to deduce some corollaries of Theorem 5.1. To illustrate this fact we notice that $p_1 = 0$ belongs to M_1. In consequence, the alternative (ii) is valid for every function $p \in C(\mathbb{R}/T\mathbb{Z})$ such that

$$p(t) \geqslant 0 \qquad \forall t \in \mathbb{R}, \qquad 0 < T^3 \int_0^T p(t)\, dt \leqslant 4 \left(\frac{\theta^*}{\pi}\right)^4.$$

2. When $p \in M_0$ every solution of (5.1) is C^1-unbounded. This can be proved combining a truncation argument as in [19] with the second Theorem of Massera.

When $p \in M_1$ every solution that is C^1-bounded in the future is asymptotic to the periodic solution. This can be deduced from the results in [2].

3. Similar results can be obtained for some more general quadratic equations of the kind

$$x'' + a(t)\, x + b(t)\, x^2 = p(t).$$

The main tools in the proof of Theorem 5.1 are the main theorem of this paper and degree theory. A similar combination was already employed in [20] in the study of the Ambrosetti-Prodi problem. The main difference with [20] is the required estimate on the linearized equation, that was pointwise in [20], and is now of integral type. The proof will be obtained after several intermediate results. Some of them may be of independent interest.

PROPOSITION 5.2. *Assume that*

$$T^3 \int_0^T p(t)\, dt < 64. \tag{5.5}$$

Then (5.1) *has at most two T-periodic solutions.*

We need two preliminary lemmas.

Lemma 5.3. *Let x be a T-periodic solution of* (5.1). *Then*

$$\int_0^T b(t)\,|x(t)|\,dt \leqslant \sqrt{T\int_0^T p(t)\,dt}.$$

Proof. Integrating (5.1) over a period, $\int_0^T bx^2 = \int_0^T p$. Combining this identity with Cauchy–Schwarz inequality,

$$\int_0^T b\,|x| \leqslant \left(\int_0^T b\right)^{1/2} \left(\int_0^T bx^2\right)^{1/2} = T^{1/2}\left(\int_0^T p\right)^{1/2}.$$

Lemma 5.4. (i) *Assume that $a \in C(\mathbb{R}/T\mathbb{Z})$, $T\int_0^T a^+ < 16$. Then every T-periodic solution of $y'' + a(t)\,y = 0$ has a constant sign or is identically zero.*

(ii) *Assume that $a_1, a_2 \in C(\mathbb{R}/T\mathbb{Z})$, $a_1 \leqslant a_2$, $T\int_0^T a_2^+ < 16$. Then, the equations $y'' + a_i(t)\,y = 0$, $i = 1, 2$, cannot have nontrivial T-periodic solutions simultaneously.*

Proof. (i) It is a consequence of Lemma 3.4 for $m = 2$.

(ii) It is analogous to the proof of Lemma 2.2.b in [19].

Proof of Proposition 5.2. First we prove that the set of T-periodic solutions of (5.1) is totally ordered; that is, given T-periodic solutions x_1, x_2,

$$x_1(t) < x_2(t) \quad \forall t \in \mathbb{R} \quad \text{or} \quad x_2(t) < x_1(t) \quad \forall t \in \mathbb{R}.$$

In fact, the difference $y = x_1 - x_2$ satisfies

$$y'' + (x_1(t) + x_2(t))\,b(t)\,y = 0$$

and, from lemma 5.3 and (5.5),

$$\int_0^T b(x_1 + x_2)^+ \leqslant 2\sqrt{T\int_0^T p} < \frac{16}{T}.$$

It follows from Lemma 5.4 that y has a constant sign and this proves that the solutions are ordered. We now prove the result by a contradiction argument. Assume that x_1, x_2, x_3 are three different T-periodic solutions of (5.1). We can assume

$$x_1(t) < x_2(t) < x_3(t) \qquad \forall t \in \mathbb{R}.$$

Define $y_1 = x_2 - x_1$, $y_2 = x_3 - x_2$. They are T-periodic and satisfy $y'' + a_i(t)\,y = 0$ with $a_1 = (x_2 + x_1)\,b$, $a_2 = (x_3 + x_2)\,b$. Also, $a_1 < a_2$ and, repeating a previous reasoning, $T\int_0^T a_2^+ < 16$. This is a contradiction with Lemma 5.4(ii).

PROPOSITION 5.5. *Assume that*

$$T^3 \int_0^T p(t)\,dt < 4. \tag{5.6}$$

(i) *If* (5.1) *has exactly one T-periodic solution then it is parabolic and unstable.*

(ii) *If* (5.1) *has exactly two T-periodic solutions then one of them is elliptic and another hyperbolic.*

To prove this result we need two lemmas. The first lemma is inspired by [19] and uses the topological index of a periodic solution, denoted by $\gamma_T(\theta)$ (see [22] for more details). The second lemma is a maximum principle for the periodic problem.

LEMMA 5.6. *Assume that* $g \in C^{0,1}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R})$ *and let* θ *be a non-degenerate T-periodic solution of*

$$x'' + g(t, x) = 0.$$

In addition assume that

$$T \int_0^T g_x(t, \theta(t))^+\,dt < 4.$$

Then $\gamma_T(\theta) = 1$ (*resp.* $\gamma_T(\theta) = -1$) *if and only if* θ *is elliptic* (*resp. hyperbolic*).

Proof. It follows from Lemma 3.4 that the distance between two consecutive zeros of a solution of the linearized equation

$$y'' + g_x(t, \theta(t))\,y = 0$$

is strictly greater than T. In consequence, the Floquet multipliers of this equation canot be negative and the Lemma is proved in the same way as Theorem 1.1 in [19].

LEMMA 5.7. *Let* $a, q \in (\mathbb{R}/T\mathbb{Z})$ *be such that* $T \int_0^T a^+(t)\,dt < 4$, $q \geqslant 0$ *and let* y *be a T-periodic solution of* $y'' + a(t)\,y = q(t)$. *Then* $y(t) \neq 0$ $\forall t \in \mathbb{R}$.

Proof. We shall prove that if τ is a zero of y then $y'(\tau) > 0$. Since y is periodic it is clear that no zero can exist.

Let ϕ be the solution of $y'' + a(t)\, y = 0$ with $\phi(\tau) = 0$, $\phi'(\tau) = 1$. As in the previous lemma we know that ϕ is positive in $(\tau, \tau + T]$. Multipliying the non-homogeneous equation by ϕ and integrating by parts one obtains

$$y'(\tau)\, \phi(\tau + T) = \int_{\tau}^{\tau + T} q(t)\, \phi(t)\, dt > 0,$$

so that $y'(\tau) > 0$.

Proof of Proposition 5.5. (i) The properties of the index imply that if (5.1) has a T-periodic solution with nonzero index then the equations that are close to (5.1) also have a T-periodic solution. When there is exactly one T-periodic solution the equation $x'' + b(t)\, x^2 = p(t) - \varepsilon$ has not periodic solutions for $\varepsilon > 0$. In consequence, the periodic solution has zero index and is therefore parabolic. The main result in [4] says that a stable periodic solution has index one, so that our solution is also unstable.

(ii) Let x_1, x_2 be the T-periodic solutions of (5.1). Applying Lemma 5.3 and (5.6) we deduce

$$T \int_0^T b(t)\, |x_i(t)|\, dt < 2, \qquad i = 1, 2. \tag{5.7}$$

The linearized equation at x_i *is*

$$y'' + 2b(t)\, x_i(t)\, y = 0. \tag{5.8}$$

The previous estimate shows that Lemma 5.6 is applicable and it is sufficient to prove that x_1, x_2 are nondegenerate and $\gamma_T(x_1) = 1$, $\gamma_T(x_2) = -1$. Since x_1, x_2 are ordered, Lemma 5.4(ii) implies that at least one of them is nondegenerate. In consequence, there exists $s_0 < 0$ such that $x'' + b(t)\, x^2 = p(t) + s_0$ has a T-periodic solution, say x_0, that is degenerate. The functions $y_i = x_i - x_0$, $i = 1, 2$ satisfy

$$y'' + b(t)(x_i(t) + x_0(t))\, y = -s_0.$$

Since the estimate (5.7) is also valid for $i = 0$ we deduce from Lemma 5.7 that $y_i(t) \neq 0$ $\forall t \in \mathbb{R}$. Now we compare (5.8) with $y'' + 2b(t)\, x_0(t)\, y = 0$ and apply Lemma 5.4(ii) to conclude that (5.8) has not T-periodic solutions different from zero. In consequence, both solutions x_i are nondegenerate.

To compute the index we notice that the additivity of degree and [5] imply that

$$\gamma_T(x_1) + \gamma_T(x_2) = 0.$$

Since x_i is nondegenerate, $|\gamma_T(x_i)| = 1$. Thus, one of the indexes must be $+1$ and the other -1 and Lemma 5.6 can be applied.

Proof of Theorem 5.1. After the previous propositions all we have to prove is that in case (ii) the elliptic solution is of twist type. Let x_1 be an elliptic solution of (5.1). The change of variables $x = z + x_1$ tranforms (5.1) to

$$z'' + 2b(t)\, x_1(t)\, z + b(t)\, z^2 = 0.$$

It follows from (5.4) and Lemma 5.3 that the Corollary 3.3(iii) is applicable and therefore $z = 0$ is of twist type.

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